

Meson Masses in High Density QCD

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Abstract

The low-energy effective theories for the two- and three-flavor color-superconductors arising in the high density limit of QCD are discussed. Using an effective field theory to describe quarks near the fermi surface, we compute the masses of the pseudo-Goldstone bosons that dominate the low-momentum dynamics of these systems.

I. INTRODUCTION

Recent developments have reinvigorated efforts to understand QCD at very high baryon density [1]- [18]. For special combinations of quark colors and flavors it is likely that a superconducting gap breaks color and flavor symmetries in interesting ways. Although the symmetry breaking is nonperturbative, it occurs when QCD is weakly coupled, and therefore perturbative QCD (pQCD) can be used to derive properties of the superconducting phase. One can conceive of scenarios in nature where it may be important to understand the behavior of QCD at high density: for instance, in neutron stars and less likely, in high energy heavy ion collisions.

Below the weak scale the standard model has the exact local gauge symmetries $SU(3)_c \otimes U(1)_{\text{em}}$ which describe the strong and electromagnetic interactions. In addition there is the exact global symmetry $U(1)_B$ corresponding to the conservation of baryon number and the approximate global symmetries $SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_A$ for a theory with $N_f = 2, 3$ light quarks. These global symmetries are broken by the quark masses and in addition the $U(1)_A$ symmetry is also broken by the strong anomaly. At extremely high density, the contribution from the anomaly is suppressed by powers of the chemical potential, μ , and $U(1)_A$ is broken only by the light quark masses. When a color-superconductor forms via the spontaneous breaking of color and flavor symmetries, there will be pseudo-Goldstone bosons that contribute to or determine the very low energy dynamics of such a system. For sufficiently low energies it is clear that an effective field theory description of these dynamics can be constructed, and will prove useful in computing contributions to observables from the far-infrared region of the theory. In recent work by Son and Stephanov [13] and by Casalbuoni and Gatto [14] the masses and decay constants of the pseudo-Goldstone bosons in the $N_f = 3$ color-flavor scenario were computed in the large- μ limit. It was found that the masses become independent of μ , while the decay constants depend linearly on μ . Later work in [12] [15] and [16] claim that the masses actually vanish in the large- μ limit and are proportional to $\sim \Delta^2/\mu^2 \log(\Delta/\mu)$. We agree with this later claim [12,15,16] and, through the use of a hierarchy of effective theories, compute the leading contribution to the meson masses and decay constants.

II. THREE FLAVORS

If we assume that the masses of the *up*, *down* and *strange* quarks are much smaller than the scale associated with the formation and dynamics of the color-superconducting state (the gap Δ), then it is appropriate to consider a theory with three flavors of massless quarks, and include mass effects in perturbation theory. In the limit of high densities the attraction leading to the gap is given by one-gluon exchange that is attractive in the color $\bar{3}$ channel. It has been argued that the most favorable state for the system is one in which there is a formation of the “color-flavor locked” condensate

$$\langle \Psi_{Lai}^\alpha \Psi_{Lbj}^\beta \rangle = \langle \Psi_{Rai}^\alpha \Psi_{Rbj}^\beta \rangle = \Delta \epsilon^{\alpha\beta c} \epsilon_{abc} \epsilon_{ij} \quad (1)$$

(α, β, \dots are color indices, a, b, \dots are flavor and i, j, \dots are spin indices) resulting in the symmetry breaking pattern $SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_A \otimes U(1)_B \otimes U(1)_{\text{em}} \rightarrow$

$SU(3)_{c+L+R} \otimes U(1)_{\text{em}}$. The value of the gap Δ was computed in [5,10,8,11] in the high density limit and does not have the usual BCS form but is instead given by

$$\Delta = c \frac{512}{g^5} \pi^4 \left(\frac{2}{N_f} \right)^{5/2} \mu e^{-\frac{3\pi^2}{\sqrt{2}g}} \quad (2)$$

where c is a constant of order unity not yet computed. For lower densities, where pQCD does not apply, the same symmetry breaking pattern was shown to occur by assuming that all quark interactions are effectively short ranged. Throughout this work we assume that Δ is a constant, independent of energy and momentum.

As we are interested in modes close to the Fermi surface where $|p| \sim \mu \gg \Delta$ we can start by considering the dynamics of quarks and gluons in the ungapped system, where the dynamics are determined by the 3 + 1 dimensional action, \mathcal{S}_{3+1} ,

$$\mathcal{S}_{3+1} = \int dt d^3x \left[-\frac{1}{4} G_{\mu\nu}^A G^{\mu\nu A} + \bar{\psi}_\alpha^a (i\not{D} + \mu\gamma^0)_{\beta}^{\alpha} \psi_a^{\beta} + \bar{\psi}_{L\alpha}^a \mathcal{M}_a^b \psi_{Rb}^{\alpha} + \bar{\psi}_{R\alpha}^a \mathcal{M}_a^{b\dagger} \psi_{Lb}^{\alpha} \right] \quad (3)$$

It is convenient to project onto the positive and negative energy states, ψ_+ and ψ_- respectively, and then eliminate ψ_- using the equations of motion [6], with

$$\psi = \psi_+ + \psi_- \quad , \quad \psi_{\pm} = \mathcal{P}_{\pm} \psi \quad , \quad \mathcal{P}_{\pm} = \frac{1}{2} (1 \pm \gamma_0 \gamma_k \mathbf{n}^k) \quad , \quad (4)$$

where $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. This procedure is similar to that used in the construction of heavy quark effective theory (HQET) [19]. Writing $\tilde{\mathcal{S}}_{3+1}$ in terms of the mode expansion for ψ_+ and working in spherical coordinates, the action describing the dynamics of the modes near the fermi surface, $\tilde{\mathcal{S}}_{3+1}$, is

$$\tilde{\mathcal{S}}_{3+1} = \frac{\mu^2}{\pi} \mathcal{S}_{1+1} \quad , \quad (5)$$

where \mathcal{S}_{1+1} is the action of a 1 + 1 dimensional field theory. The two-component quark field $\psi_+ = \chi_a^{\alpha}(E, k, \mathbf{n})$ depends on an energy $E \ll \mu$, a momentum $k \ll \mu$, and a unit vector pointing toward the fermi surface \mathbf{n} . As the anti-quarks have been integrated out of the theory at the scale μ corresponding to the top of the fermi surface, the effective field theory described by \mathcal{S}_{1+1} will be an expansion in terms of E/μ and k/μ , as outlined in [6].

Perturbative computations around the superconducting state can be performed by adding and subtracting a quark gap term in the QCD lagrangian. The condition that the subtracted gap term does not contribute at each order in perturbation theory is equivalent to the gap equation. The gaps for the positive and negative energy states will in general be different, and in terms of the ψ_{\pm} fields we have an additional contribution to the lagrange density of the form

$$\mathcal{L}^{\Delta} = \frac{\Delta}{2} \epsilon_{\alpha\beta I} \epsilon^{abI} \psi_{a+}^{T\alpha} C \psi_{b+}^{\beta} + \frac{\overline{\Delta}}{2} \epsilon_{\alpha\beta I} \epsilon^{abI} \psi_{a-}^{T\alpha} C \psi_{b-}^{\beta} + \text{h.c.} \quad (6)$$

The anti-gap, $\overline{\Delta}$, has not been computed at this point in time, but a recent discussion can be found in [11].

A further simplification can be made by writing the quark fields, χ , in terms of the mass eigenstates of the condensate (neglecting the quark masses)

$$\chi_a^\alpha = \frac{1}{\sqrt{2}} \sum_{A=1}^9 \chi^A (\lambda^A)_a^\alpha \quad , \quad (7)$$

where λ^A with $A = 1, \dots, 8$ are the Gell-Mann matrices and $\lambda^9 = \sqrt{2/3} I_3$. After eliminating ψ_- by the equations of motion, the part of the leading order action that does not depend upon the quark masses is

$$\mathcal{S}_{1+1}^{(0)} = \sum_{A=1}^9 \int \frac{d\mathbf{n}}{4\pi} \frac{dE}{(2\pi)^2} \frac{dk}{(2\pi)^2} \left[\chi_{\mathbf{n}}^{A\dagger} [E - k] \chi_{\mathbf{n}}^A - \frac{\Delta^A}{2} (\chi_{\mathbf{n}}^{AT} C \chi_{-\mathbf{n}}^A + \text{h.c.}) + \dots \right] \quad , \quad (8)$$

where $\Delta^A = \Delta$ for $A = 1, \dots, 8$, while $\Delta^9 = -2\Delta$. Interactions between the χ^A and the gauge fields have not been shown. The ellipses denote operators that are suppressed by powers of μ . As we are assuming that the quark masses are small compared to the scales associated with formation of the superconducting state, we can treat the quark masses in perturbation theory. The leading order contributions from the quark masses are described by the action

$$\begin{aligned} \mathcal{S}_{1+1}^{(m^2)} = & - \sum_{A,B=1}^9 \int \frac{d\mathbf{n}}{4\pi} \frac{dE}{(2\pi)^2} \frac{dk}{(2\pi)^2} \left[\frac{1}{4\mu} \chi_{\mathbf{n}}^{A\dagger} \chi_{\mathbf{n}}^B \text{Tr} [\lambda^A \mathcal{M}^\dagger \mathcal{M} \lambda^B] \right. \\ & \left. + \frac{\overline{\Delta}}{16\mu^2} [(\chi_{R\mathbf{n}}^{AT} C \chi_{R-\mathbf{n}}^B + \chi_{L\mathbf{n}}^{A\dagger} C \chi_{L-\mathbf{n}}^{B*}) \mathcal{Y}^{AB} + \text{h.c.}] \right] \quad , \quad (9) \end{aligned}$$

where

$$\mathcal{Y}^{AB} = \text{Tr} [\lambda^A \mathcal{M} \lambda^B \mathcal{M}] - \text{Tr} [\lambda^A \mathcal{M}] \text{Tr} [\lambda^B \mathcal{M}] \quad , \quad (10)$$

and where the L, R subscripts on χ^A denotes the helicity/chirality state. The appearance of explicit factors of $1/\mu$ associated with the mass terms is no surprise, in fact, the first term in Eq. (9) follows naturally from the expansion of $E = \sqrt{p^2 + m^2} = p + m^2/(2p) + \dots$ for $p \sim \mu \gg m$. Contributions from more insertions of the light quark mass matrix or from higher derivative operators are suppressed by powers of μ . The actions in Eq. (8) and Eq. (9) describe the dynamics of modes near the fermi surface. Contributions to observables arising from modes far from the fermi surface are suppressed by powers of μ , and enter through the higher dimension operators that we have not shown. An important point is that this lagrange density involves only analytic functions of the light quark masses and the gap Δ . For very low-energy dynamics of the system $|\mathbf{p}| \ll \Delta$ it is appropriate to construct an effective field theory for the pseudo-Goldstone modes alone. This lagrange density will involve analytic functions of the light quark masses, but it will have non-analytic dependence on Δ . This nonanalytic dependence can be computed from the effective theory describing the momentum region $\Delta \ll |\mathbf{p}| \ll \mu$, as described above. This is in direct analogy with the nonanalytic contributions to observables in the light meson sector, such as terms of the form $\log(m_\pi^2/\Lambda_\chi^2)$ or $\sqrt{m_q}$.

For momenta much below the gap, $p \ll \Delta$, the relevant degrees of freedom are the nine pseudo-Goldstone bosons resulting from symmetry breaking due to the condensate. They are described by the fields

$$\Sigma = e^{i2M/f_8} \quad , \quad V = e^{i2\eta_1/f_1} \quad , \quad (11)$$

with

$$M = \begin{pmatrix} \pi_3/\sqrt{2} + \eta_8/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi_3/\sqrt{2} + \eta_8/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta_8/\sqrt{6} \end{pmatrix} . \quad (12)$$

These fields and the quark mass matrix transform under $SU(3)_L \times SU(3)_R \times U(1)_B \times U(1)_A$ as

$$\Sigma \rightarrow L\Sigma R^\dagger , \quad V \rightarrow e^{-i4\beta}V , \quad \mathcal{M} \rightarrow e^{+i2\beta}L\mathcal{M}R^\dagger , \quad (13)$$

where β is the $U(1)_A$ phase. The interactions of these mesonic excitations will have an expansion around zero momentum described by

$$\begin{aligned} \mathcal{L} = & \frac{f_8^2}{8} \text{Tr} [\partial_0 \Sigma \partial_0 \Sigma^\dagger] - |\mathbf{v}_8|^2 \frac{f_8^2}{8} \text{Tr} [\nabla_i \Sigma \nabla_i \Sigma^\dagger] + \frac{f_1^2}{8} \partial_0 V \partial_0 V^* - |\mathbf{v}_1|^2 \frac{f_1^2}{8} \nabla_i V \nabla_i V^* \\ & + A_1 \left(\text{Tr} [\mathcal{M}^\dagger \Sigma] \text{Tr} [\mathcal{M}^\dagger \Sigma] V^* + \text{h.c.} \right) + A_2 \left(\text{Tr} [\mathcal{M}^\dagger \Sigma \mathcal{M}^\dagger \Sigma] V^* + \text{h.c.} \right) \\ & + A_3 \text{Tr} [\mathcal{M}^\dagger \Sigma] \text{Tr} [\mathcal{M} \Sigma^\dagger] + A_4 \text{Tr} [\mathcal{M} \mathcal{M}^\dagger] + \dots , \end{aligned} \quad (14)$$

where the ellipses denote operators suppressed by powers of p/Δ and \mathcal{M}/Δ (for a recent discussion of this see [18]). The velocities of the octet and singlet bosons are \mathbf{v}_8 and \mathbf{v}_1 respectively. The coefficients appearing in Eq.(14) are determined by matching S-matrix elements in the theory above Δ with those in the theory below Δ , determined from Eqs.(8), (9) and Eq. (14) respectively. The form of the mass terms is exactly the same as the second order terms appearing in the Lagrange density written down by Gasser and Leutwyler [20].

A. Decay Constants

The decay constants f_8 and f_1 can be found by computing the Debye and Meissner masses of fictitious gauge bosons coupled to currents in both pQCD [13] and the low-energy effective field theory, Eq. (14). In the low-energy regime the two calculations must produce identical results, thereby determining f_8 and f_1 order by order in the chiral, $1/\mu$ and α_s expansions.

Gauging the $U(1)_A$ axial current leads to $D_\mu = \partial_\mu + ieW_\mu Q_1$, where Q_1 is the axial charge operator, $Q_1 \Psi_L = +1\Psi_L$, $Q_1 \Psi_R = -1\Psi_R$, $Q_1 V = -4V$ and $Q_1 \tilde{\Sigma} = 0$. It is then straightforward to show that the mass of the W_μ fields at leading order in the low-energy effective field theory are

$$\Delta\mathcal{L} = 2 e^2 f_1^2 \left[W_0^2 - |\mathbf{v}_1|^2 W_i^2 \right] . \quad (15)$$

Computing these masses in pQCD through Eq. (8) one finds that the Debye mass is given by the two graphs shown in Fig. (1) with the result

$$\Delta\mathcal{L}_{\text{QCD}} = e^2 \frac{9\mu^2}{2\pi^2} W_0^2 , \quad (16)$$

and is essentially identical to the standard many-body calculations described in [21]. In order to reproduce this with the effective theory

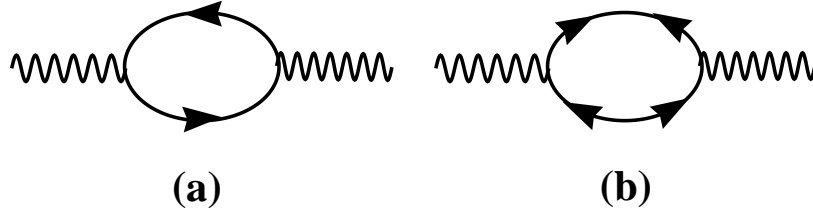


FIG. 1. One loop diagrams that give the leading contribution to the meson decay constants.

$$f_1 = \frac{9}{4} \frac{\mu^2}{\pi^2} \quad , \quad (17)$$

which agrees with [13]. To compute the speed of the Goldstone mode, we need to compute the Meissner mass in both theories. In addition to the diagrams of Fig. (1) (with different couplings) there is a contribution from a counterterm. The sum of diagrams yields a vanishing Meissner mass in the normal phase and a non-zero mass in the superconducting phase, see [21]. The speed is found to be $|\mathbf{v}_1|^2 = \frac{1}{3}$, in agreement with [13]. As discussed in [13] the analogous calculation for the baryon number Goldstone boson is identical at leading order, $|\mathbf{v}_H|^2 = |\mathbf{v}_1|^2 = \frac{1}{3}$.

To determine the decay constant of the octet pseudo-Goldstone bosons, f_8 , the Debye and Meissner masses of fictitious gauge bosons associated with octet currents are computed. We find

$$f_8^2 = \frac{\mu^2}{\pi^2} \frac{21 - 8 \log[2]}{9} \quad , \quad (18)$$

which differs by a factor of 2 from Ref. [13] after differences due to the definition of f_8 are taken into account. The pion velocity is found to be $|\mathbf{v}_8|^2 = \frac{1}{3}$ which agrees with Ref. [13]. Therefore, all Goldstone modes have the same speed at leading order in the expansion.

As the decay constants scale like $f \sim \mu$, it is apparent that the dynamics of the pseudo-Goldstone bosons do not receive significant contributions from loops in the low-energy effective theory of Eq. (14). The naive size of the counterterms is set by $1/\Delta$, while the contribution from loops is set by $1/\mu$. Therefore, once the coefficients in the low energy effective theory have been determined, only tree-level diagrams need to be considered.

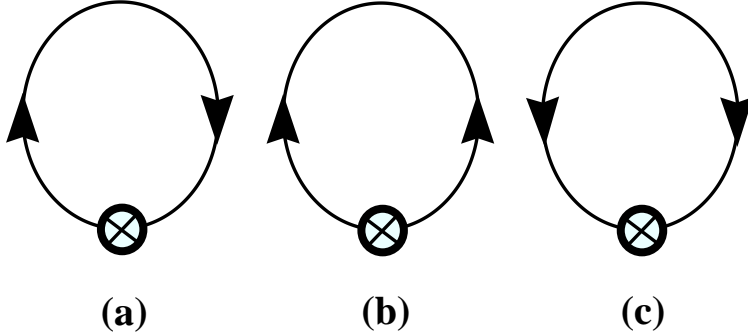


FIG. 2. One loop diagrams that give the leading contribution to the meson masses. Graph (a) contributes only to A_4 which does not impact the dynamics of the pseudo-Goldstone modes, while graphs (b) and (c) contribute to both A_1 and A_2 . The crossed circle denotes an insertion of the quark mass operators in the effective field theory defined in Eq. (9).

B. Meson Masses

The coefficients A_i in Eq. (14) can be determined by matching the change in the ground state energy due to the quark masses in the high (Eqs. (8) and (9)) and low energy theories (Eq. (14)). In the low energy theory, the change in energy density can be easily determined from Eq. (14) by setting $\Sigma = V I_3 = I_3$, to yield

$$\begin{aligned} \delta\mathcal{E} = & A_1 \left((\text{Tr} [\mathcal{M}])^2 + \text{h.c.} \right) + A_2 \left(\text{Tr} [\mathcal{M}^2] + \text{h.c.} \right) \\ & + A_3 \text{Tr} [\mathcal{M}] \text{Tr} [\mathcal{M}^\dagger] + A_4 \text{Tr} [\mathcal{M} \mathcal{M}^\dagger] \quad . \end{aligned} \quad (19)$$

The operator with coefficient A_4 does not contribute to the dynamics of the pseudo-Goldstone modes, and we do not calculate it.

Computation of the energy density in the 3+1 dimensional high energy theory can easily be done, by noting that the action of the 3+1 dimensional theory is a factor of μ^2/π times the action of the 1+1 dimensional theory. Thus the energy density in the 3+1 dimensional theory is μ^2/π times the energy density computed in the 1+1 dimensional theory. We use dimensional regularization and minimal subtraction to define divergent integrals that occur at loop level in the 1+1 dimensional theory. The shift in the vacuum energy due to the light quark masses results from the tadpole diagrams shown in Fig. (2), where the vertex arises from Eq. (9). At leading order we find

$$\delta\mathcal{E}^{\text{loop}} = \frac{\overline{\Delta}\Delta}{4\pi^2} \log\left(\frac{\Delta}{\Lambda}\right) \left((\text{Tr}[\mathcal{M}])^2 - \text{Tr}[\mathcal{M}^2] \right) + \text{h.c.} , \quad (20)$$

where Λ is the renormalization scale and we have only shown the term nonanalytic in Δ/Λ . The form of our expression agrees with the results of [12,15,16]. The explicit dependence on Λ shown in Eq. (20) is absorbed by an equal but opposite Λ dependent counterterm that, for $\Lambda \sim \mu$ generates a shift in energy $\sim \overline{\Delta}\Delta/4\pi^2$. This counterterm contribution is suppressed compared to the contribution in Eq.(20) by the large $\log(\Delta/\mu)$ factor and will be neglected. Matching this result with the corresponding shift in energy computed in the effective theory we find

$$A_1 = -A_2 = -\frac{\overline{\Delta}\Delta}{4\pi^2} \log\left(\frac{\Delta}{\mu}\right) = A , \quad A_3 = 0 . \quad (21)$$

The meson masses at leading order are found by expanding Eq.(14) to second order in the meson fields. The charged meson masses are

$$m_{\pi^+}^2 = \frac{8A}{f_8^2} (m_u + m_d)m_s , \quad m_{K^+}^2 = \frac{8A}{f_8^2} (m_u + m_s)m_d , \quad m_{K^0}^2 = \frac{8A}{f_8^2} (m_d + m_s)m_u , \quad (22)$$

and the neutral meson mass matrix is

$$\begin{aligned} m_{33}^2 &= \frac{8A}{f_8^2} m_s(m_u + m_d) , \quad m_{88}^2 = \frac{8A}{3f_8^2} [m_s(m_u + m_d) + 4m_um_d] \\ m_{11}^2 &= \frac{16A}{f_1^2} [m_s(m_u + m_d) + m_um_d] , \quad m_{13}^2 = -\frac{8A\sqrt{2}}{f_1 f_8} (m_u - m_d)m_s \\ m_{18}^2 &= \frac{16A}{\sqrt{6}f_1 f_8} [m_s(m_u + m_d) - 2m_um_d] , \quad m_{38}^2 = -\frac{8A}{\sqrt{3}f_8^2} (m_u - m_d)m_s . \end{aligned} \quad (23)$$

In the limit of zero mixing $m_{11} = m_{\eta'}$, $m_{33} = m_{\pi^0}$ and $m_{88} = m_{\eta}$. The eigenvectors of this matrix are close to the quark flavor eigenstates for values of the quark masses consistent with standard chiral perturbation theory [20].

It is interesting to consider mass relations between the mesons. At this order in perturbation theory there is a relation between the meson masses without any assumption about the light quark mass hierarchy,

$$m_{\eta'}^2 + m_{\pi^0}^2 + m_{\eta}^2 = \left(\frac{2}{3} + \frac{f_8^2}{f_1^2} \right) [m_{\pi^+}^2 + m_{K^+}^2 + m_{K^0}^2] , \quad (24)$$

where m_{η} , $m_{\eta'}$ and m_{π^0} are the mass eigenvalues found by diagonalizing the neutral meson mass matrix. Further, if one assumes $m_{u,d}/m_s \ll 1$ and neglects such terms,

$$\frac{m_{K^+}^2}{m_{\pi^+}^2} = \frac{m_d}{m_d + m_u} , \quad \frac{m_{K^0}^2}{m_{\pi^+}^2} = \frac{m_u}{m_d + m_u} , \quad (25)$$

and hence $m_{K^+}^2 + m_{K^0}^2 = m_{\pi^+}^2$. It is also clear that there is an inverse mass hierarchy, e.g. $m_{\pi^+} > m_{K^+} > m_{K^0}$.

In addition to corrections to the coefficients A_1 and A_2 arising at higher orders in $1/\mu$ and α_s there will be contributions to the A_3 , such as those shown in Fig. (3). The operator in the effective theory appearing in Fig. (3) results from a two loop diagram in QCD where two of the propagators are far off-shell, resulting in an effectively local vertex for momenta much less than μ .

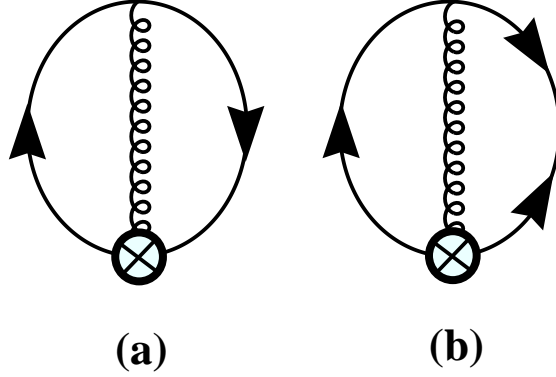


FIG. 3. Some of the two loop diagrams that contribute to A_3 . The crossed circle denotes an insertion of a quark mass operator involving the gluon field that arises at higher orders in the effective field theory expansion. The curly lines denote gluons.

III. TWO FLAVORS

In the two flavor case (three colors) the most favored condensate is

$$\langle \Psi_{Lai}^\alpha \Psi_{Lbj}^\beta \rangle = \langle \Psi_{Rai}^\alpha \Psi_{Rbj}^\beta \rangle = \Delta \epsilon^{\alpha\beta 3} \epsilon_{ab} \epsilon_{ij} \quad (26)$$

which breaks $SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_A \otimes U(1)_B \otimes U(1)_{em}$ down to $SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{\tilde{em}} \otimes U(1)_{\tilde{B}}$, where $U(1)_{\tilde{em}}$ and $U(1)_{\tilde{B}}$ are linear combinations of electromagnetism, baryon number and the eighth gluon, A_μ^8 [3,4]. At energies below the gap the dynamical degrees of freedom are three massless gluons, two ungapped quarks and the pseudo-Goldstone boson associated with spontaneous breaking of $U(1)_A$. In contrast to the three flavor case, there is no pseudo-Goldstone boson associated with the breaking of baryon number. The Goldstone field describing the η' meson related to the breaking of $U(1)_A$ can be parametrized as

$$V = e^{i2\eta'/f_{\eta'}} \quad , \quad V \rightarrow e^{-i4\beta} V \quad . \quad (27)$$

The effective lagrangian describing the dynamics of the η' at leading order in the ∂/Δ and m/Δ expansion is

$$\mathcal{L} = \frac{f_{\eta'}^2}{8} \left(D_0 V D_0 V^* - |\mathbf{v}|^2 D_i V D_i V^* \right) + B \left[\det(\mathcal{M}) V + \text{h.c.} \right] . \quad (28)$$

Using the diagonal basis for the gapped quark fields with $a, i = 1, 2$

$$\Psi_i^a = \frac{1}{\sqrt{2}} \sum_{A=1}^4 \left(\tau^A \right)_i^a \Psi^A \quad (29)$$

where τ^A are Pauli matrices for $A = 1, 2, 3$ and $\tau^4 = I_2$, the gaps for the four gapped quarks are $-\Delta_4 = \Delta_A = \Delta$ for $A = 1, 2, 3$, and the others remain ungapped. Calculation of the decay constant and vacuum energy shift due to the light quark masses is similar to that performed in the three flavor case and gives

$$f_{\eta'}^2 = \frac{\mu^2}{\pi^2} , \quad |\mathbf{v}|^2 = \frac{1}{3} , \quad B = -\frac{\overline{\Delta}\Delta}{2\pi^2} \log\left(\frac{\Delta}{\mu}\right) \quad (30)$$

and therefore an η' mass of

$$m_{\eta'}^2 = \frac{8B_1 m_u m_d}{f_{\eta'}^2} = -\frac{4\overline{\Delta}\Delta}{\mu^2} \log\left(\frac{\Delta}{\mu}\right) m_u m_d , \quad (31)$$

where we have neglected the contribution from the local counterterm to the η' mass.

The ungapped quarks interact with themselves at leading order via the exchange of a massive gluon, inducing a four-quark operator for scales below $g_s \mu$. The coefficient of this operator is independent of the strong coupling constant α_s (which arises from a cancellation between the couplings and the gluon mass) and scales like $1/\mu^2$. This interaction is repulsive since the two ungapped quarks are in a **6** of color and further the interaction vanishes in the high density limit. Thus we do not expect condensation of the “green” quarks in the high density limit. It is interesting to note that the η' does not couple directly to the ungapped quarks since the η' field is an excitation of a condensate involving the “red” and “blue” colors only. Thus the axial coupling constant describing this interaction is suppressed by powers of α_s and consequently the tensor force between the ungapped quarks in the low energy theory arising from the exchange of a single η' is suppressed by α_s^2 .

Another interesting aspect of the two-flavor case is the presence of an unbroken pure $SU(2)_c$ gauge theory. The gluons associated with this gauge group do not interact with the ungapped quarks. The confinement scale of this theory can be estimated by assuming the only modification to the evolution of the strong coupling arises from the particle content. This provides an estimate of the scale at which the theory becomes strongly coupled,

$$\mu_2^{\text{conf.}} = \mu_3^{\text{conf.}} \left(\frac{\Delta}{\Lambda_{\text{QCD}}} \right)^{-\frac{7}{22}} e^{6\pi(\frac{1}{22}-\frac{1}{29})} , \quad (32)$$

where $\mu_{2,3}^{\text{conf.}}$ are these scales in the two and three color theories respectively. For large Δ , this scale is much lower than Λ_{QCD} , but for reasonable values of Δ , $\mu_2^{\text{conf.}} \sim \mu_3^{\text{conf.}}$. This suggests that pure Yang-Mills glueballs will appear in the low energy theory with masses of order $\sim \Lambda_{\text{QCD}}$.

IV. CONCLUSION

We have examined the high density limit of QCD where there are two and three flavors of “light” quarks below the scale relevant to the formation of a color superconducting state. Using an effective field theory to describe quark modes near the Fermi surface we have determined the decay constants and masses of the pseudo-Goldstone bosons that arise in each theory at leading order in the $1/\mu$ expansion. The masses of these pseudo-Goldstone modes vanish in the high density limit. In order to determine the behavior of these systems at a moderate density the subleading corrections (e.g. $1/\mu$, α_s) need to be determined, including the contributions from instantons [16].

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